Ion acoustic waves in a thin radially inhomogeneous plasma column

1978 J. Phys. A: Math. Gen. 11 L63

(http://iopscience.iop.org/0305-4470/11/3/005)

J. Phys. A: Math. Gen., Vol. 11, No. 3, 1978. Printed in Great Britain

LETTER TO THE EDITOR

Ion acoustic waves in a thin radially inhomogeneous plasma column

E Mateev† and I Zhelyazkov‡§

†Faculty of Physics, Sofia University, 1126 Sofia, Bulgaria
‡Institut für Theoretische Physik, Ruhr-Universität Bochum, 4630 Bochum, Federal Republic of Germany

Received 1 November 1977

Abstract. The dispersion of low-frequency electrostatic waves propagating along a radially inhomogeneous plasma cylinder has been investigated. It is shown that the density inhomogeneity introduces only an additional collisionless damping. The latter depends on the integral and differential characteristics of the density profile.

1. Introduction

In connection with plasma diagnostics via laser light scattering and RF heating of plasmas through parametric processes, the spectra of plasma waves in finite plasma systems have been studied. It is well known that besides considering the bulk modes one should also account for the so called surface modes, i.e. the waves whose fields are localised in the region of the plasma–dielectric boundary (Krall and Trivelpiece 1973). The spectral characteristics of low-frequency surface waves propagating in a thin homogeneous plasma column were investigated by means of both the fluid (Petrova *et al* 1975) and the kinetic (Shivarova and Zhelyazkov 1978) plasma models. However, in most experiments the plasma is highly inhomogeneous at least in one (radial) direction. In this Letter, we accordingly study the influence of plasma–density inhomogeneity on the spectrum and the damping rate of long-wavelength low-frequency waves propagating in a radially inhomogeneous plasma column.

2. Basic equations and dispersion relation

Let us consider the propagation of axially symmetric low-frequency ion acoustic waves of the form $g(r)\exp(-\mathrm{i}\omega t+\mathrm{i}kz)$ along the z axis of a radially inhomogeneous cylindrical warm plasma bounded by dielectric (vacuum). We assume that the density profile has the form shown in figure 1. The only condition which the profile must satisfy is

$$\int_0^\infty rn_0(r)\,\mathrm{d}r=r_0^2n_0(0)<\infty.$$

§ On leave of absence from Faculty of Physics, Sofia University, 1126 Sofia, Bulgaria.

0305-4770/78/0003-9063 $01.00

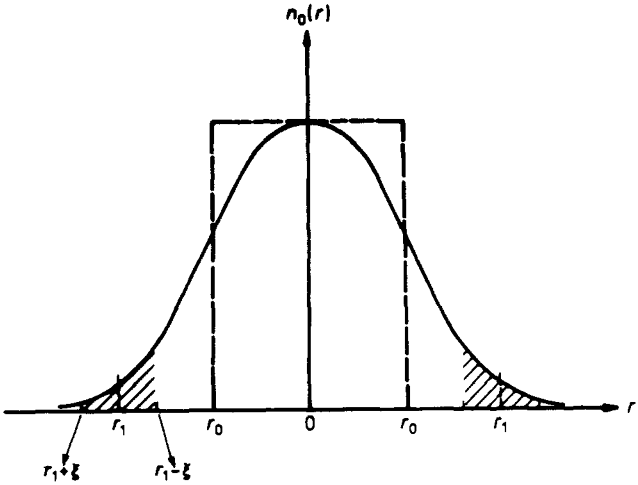

Figure 1. Density profile of the plasma column.

This equality defines the effective plasma radius r_0 (figure 1). The basic equations which govern the wave field potential and plasma particles' dynamics (after Fourier analysing in time) are:

$$-\mathrm{i}\omega n_\mathrm{i} + \nabla \cdot (n_0(r)\boldsymbol{v}_\mathrm{i}) = 0, \tag{1}$$

$$(-\mathrm{i}\omega + \nu_\mathrm{i})\boldsymbol{v}_\mathrm{i} = \frac{e}{M}\nabla\phi, \tag{2}$$

$$0 = \frac{e}{m}\nabla\left(\phi - \frac{T_\mathrm{e}}{en_0(r)} n_\mathrm{e}\right), \tag{3}$$

$$\Delta\phi = 4\pi e(n_\mathrm{e} - n_\mathrm{i}), \tag{4}$$

where n_α and $\boldsymbol{v}_\alpha$ are the number densities and fluid velocities of the particles of species $\alpha(\alpha = \mathrm{e}, \mathrm{i})$, T_e is the electron temperature (for convenience ions are assumed to be cold, $T_\mathrm{i} = 0$), ν_i is the ion collision frequency. The other notation is standard. By eliminating n_i, n_e and $\boldsymbol{v}_\mathrm{i}$ from the set (1)–(4), we obtain

$$\nabla \cdot \left[\left(1 - \frac{\omega_{\mathrm{pi}}^2(r)}{\omega(\omega + \mathrm{i}\nu_\mathrm{i})}\right)\nabla\phi\right] - \frac{\omega_{\mathrm{pi}}^2(r)}{v_\mathrm{s}^2} = 0, \tag{5}$$

where $\omega_{\mathrm{pi}}(r) = (4\pi n_0(r)e^2/M)^{1/2}$ is the ion plasma frequency and $v_\mathrm{s} = (T_\mathrm{e}/M)^{1/2}$ is the speed of sound.

Since we consider the propagation of long-wavelength ion acoustic waves in a thin cylinder, we take $kr \ll 1$. Moreover, because of density inhomogeneity, there exists a surface with radius r_1 (figure 1) where the wave frequency equals the local ion plasma frequency, i.e.

$$\omega^2 \equiv \omega_{\mathrm{pi}}^2(r).$$

In the vicinity of r_1 ($r \in (r_1 - \xi, r_1 + \xi)$), the ion plasma frequency can be expressed as

$$\omega_{\mathrm{pi}}^2(r) = \omega^2[1 - a(r - r_1)], \tag{6}$$

where $a = -n_{01}^{-1}\,\mathrm{d}n_0/\mathrm{d}r > 0$ (n_{01} is the equilibrium number density corresponding to r_1) characterises the density gradient around r_1.

If the density profile is such that $\omega_{\mathrm{pi}}^2(r_1+\xi) \ll \omega^2$, for $r > r_1$, equation (5) reads

$$\Delta\phi = 0$$

with a solution (bounded at $r\to\infty$)

$$\phi(r) = \Phi_\infty K_0(kr),$$

where K_0 is the modified Bessel function of the second kind and zeroth order. For $kr \ll 1$, the above expression for ϕ may be written in the form:

$$\phi(r) \simeq \Phi_\infty[\ln(p) - \ln(kr)], \tag{7}$$

where $p = 2/e^C \geqslant 1$, and $C = 0{\cdot}577$ is Euler's constant.

When $r \leqslant r_1$, we can rewrite equation (5) as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(1-\frac{\omega_{\mathrm{pi}}^2(r)}{\omega(\omega+\mathrm{i}\nu_\mathrm{i})}\right)\frac{\partial\phi}{\partial r}\right] = \left[k^2\left(1-\frac{\omega_{\mathrm{pi}}^2(r)}{\omega(\omega+\mathrm{i}\nu_\mathrm{i})}\right)+\frac{\omega_{\mathrm{pi}}^2(r)}{v_\mathrm{s}^2}\right]\phi. \tag{8}$$

Because the quantities on the right-hand side of (8) are small ($\sim k^2r^2$) with respect to those on the left-hand side, we may seek the solution of equation (8) in the form

$$\phi = \phi_0 + \phi_1 \qquad \text{with } \phi_1 \ll \phi_0,$$

ϕ_0 is the solution of (8) when its right-hand side is zero. Applying the usual iterative procedure, and using the conditions

$$\phi_1(0) = 0 \qquad \text{and} \qquad \left.\frac{\partial\phi_1}{\partial r}\right|_{r=0} = 0,$$

we obtain

$$\phi(r) = \Phi_0\left\{1 + \int_0^r \frac{\mathrm{d}r'}{r'[\omega(\omega+\mathrm{i}\nu_\mathrm{i}) - \omega_{\mathrm{pi}}^2(r')]}\right.$$
$$\left.\times\int_0^{r'} \mathrm{d}r''r''\left[k^2\omega(\omega+\mathrm{i}\nu_\mathrm{i}) - \left(k^2 - \frac{\omega(\omega+\mathrm{i}\nu_\mathrm{i})}{v_\mathrm{s}^2}\right)\omega_{\mathrm{pi}}^2(r'')\right]\right\}. \tag{9}$$

It is obvious that the two solutions (7) and (9) have a common area of validity near r_1. In that region we may neglect in (9) the integrals which are of the order k^2r^2 with respect to $\ln(kr)$ and the free terms The result is

$$\phi(r) \simeq \Phi_0\left[1 - \left(k^2 - \frac{\omega(\omega+\mathrm{i}\nu_\mathrm{i})}{v_\mathrm{s}^2}\right)\left(\int_0^{r_1-\xi}\frac{\mathrm{d}r'}{r'[\omega(\omega+\mathrm{i}\nu_\mathrm{i}) - \omega_{\mathrm{pi}}^2(r')]}\int_0^{r'}\mathrm{d}r''r''\omega_{\mathrm{pi}}^2(r'')\right.\right.$$
$$+\int_{r_1-\xi}^{r_1+\xi}\frac{\mathrm{d}r'}{r'[\omega(\omega+\mathrm{i}\nu_\mathrm{i}) - \omega_{\mathrm{pi}}^2(r')]}\int_0^{r'}\mathrm{d}r''r''\omega_{\mathrm{pi}}^2(r'')$$
$$\left.\left.+\frac{1}{\omega(\omega+\mathrm{i}\nu_\mathrm{i})}\int_{r_1+\xi}^{r}\frac{\mathrm{d}r'}{r'}\int_0^{r'}\mathrm{d}r''r''\omega_{\mathrm{pi}}^2(r'')\right)\right]. \tag{9a}$$

We note that all the integrals $\int_0^{r'}\mathrm{d}r''r''\omega_{\mathrm{pi}}^2(r'')$ can be replaced by

$$\int_0^{r'\to\infty}\mathrm{d}r''r''\omega_{\mathrm{pi}}^2(r'') = r_0^2\omega_{\mathrm{pi}}^2(0),$$

except for cases having very peculiar density profiles. If the density gradient near r_1 is such that $\omega_{pi}^2(r_1-\xi)\gg\omega^2$, the first integral in (9*a*) can be neglected (it is of the order $k^2r_1^2$) and the straightforward integration of the other two leads to

$$\phi(r)\approx\Phi_0\bigg[1+r_0^2\omega_{pi}^2(0)\Big(\frac{k^2}{\omega(\omega+i\nu_i)}-\frac{1}{v_s^2}\Big)\Big(\ln(kr_1)+\frac{i\pi}{ar_1}\Big)$$
$$-r_0^2\omega_{pi}^2(0)\Big(\frac{k^2}{\omega(\omega+i\nu_i)}-\frac{1}{v_s^2}\Big)\ln(kr)\bigg],\qquad(9b)$$

where (6) has been used.

Noting that solutions (7) and (9*b*) are valid in the same region, we equate the free terms and the coefficients in front of ln (kr) in (7) and (9*b*), to find

$$\Phi_0\bigg[1+r_0^2\omega_{pi}^2(0)\Big(\frac{k^2}{\omega(\omega+i\nu_i)}-\frac{1}{v_s^2}\Big)\Big(\ln(kr_1)+\frac{i\pi}{ar_1}\Big)\bigg]-\Phi_\infty\ln(p)=0$$
$$-\Phi_0r_0^2\omega_{pi}^2(0)\Big(\frac{k^2}{\omega(\omega+i\nu_i)}-\frac{1}{v_s^2}\Big)+\Phi_\infty=0.$$

The condition for solvability of this set of equations yields the following dispersion relation:

$$1-r_0^2\omega_{pi}^2(0)\Big(\frac{k^2}{\omega(\omega+i\nu_i)}-\frac{1}{v_s^2}\Big)\Big[\ln\Big(\frac{p}{kr_1}\Big)-\frac{i\pi}{ar_1}\Big]=0.\qquad(10)$$

For almost all density profiles $ar_1>1$, the dispersion relation (10) can be solved to obtain the spectrum and the damping rate of the low-frequency ion acoustic waves propagating along a thin radially inhomogeneous plasma column:

$$\omega=kv_s\Big(1-\frac{\lambda_e^2(0)}{2r_0^2\ln(p/kr_1)}\Big)-i\frac{\nu_i}{2}-i\frac{\pi kv_s\lambda_e^2(0)}{2r_0^2ar_1[\ln(p/kr_1)]^2},\qquad(11)$$

where $\lambda_e(0)\ll r_0$ is the electron Debye length at the axis of the column.

3. Concluding remarks

We emphasise the fact that the long-wavelength low-frequency waves which can propagate along a thin highly inhomogeneous plasma column (the scale length of density inhomogeneity is much smaller than the wavelength) are surface waves; their fields are localised near the surface at $r=r_1$.

In accordance with expression (11) the spectrum and the collisionless damping rate due to the density inhomogeneity depend on one integral column's characteristics (r_0) and on one differential characteristics ($a=n_{01}^{-1}\,dn_0/dr$). It is easy to see, however, that the plasma inhomogeneity does not practically affect the wave frequency since the term $\lambda_e^2(0)[2r_0^2\ln(p/kr_1)]^{-1}\ll 1$. The nature of the collisionless damping of the waves is similar to that which governs the energy dissipation of high-frequency surface waves in an inhomogeneous electronic cold plasma (Stepanov 1965, Demchenko and Zayed 1972). In our case it is associated with the resonance at the local (at r_1) ion plasma frequency. For a highly ionised plasma this damping can be larger than the collisional damping $\nu_i/2$.

For a homogeneous plasma column with sharp boundary ($ar_1 \to \infty$ and $r_0 = R/\sqrt{2}$, where R is the radius of the cylinder) equation (10) gives the spectrum

$$\omega = kv_s\left(1 + \frac{2\lambda_e^2}{R^2 \ln(p\sqrt{2}/kR)}\right)^{-1/2}. \tag{12}$$

Formula (12) agrees with the expression for ω obtained by Petrova *et al* (1975) within the accuracy of $p\sqrt{2} \sim 1$.

It is worth noting that the expression for the damping rate arising due to the plasma inhomogeneity can be checked experimentally. By using an appropriate experimental technique (for example by measuring frequency shifts of a cavity resonator (Kent *et al* 1971)), one may determine the plasma density profile and hence calculate the two quantities r_0 and a.

The spectrum of low-frequency surface waves in an inhomogeneous plasma for the case in which the wavelength of the perturbation is smaller than the scale length of plasma density inhomogeneity will be the subject of a subsequent paper.

Acknowledgments

One of us (IZh) acknowledges the Alexander von Humboldt Foundation for award of a Senior Fellowship and thanks Professor Spatschek and Drs Shukla and Yu for helpful discussions.

References

Demchenko V V and Zayed K E 1972 *Physica* **59** 385
Kent G, Sinnot D and Kent P 1971 *J. Appl. Phys.* **42** 2847
Krall N A and Trivelpiece A W 1973 *Principles of Plasma Physics* (New York: McGraw-Hill)
Petrova R, Stoychev T and Shivarova A 1975 *Acta Politech. Práce ČVUT v Praze* **4** 111
Shivarova A and Zhelyazkov I 1978 *Czech. J. Phys.* submitted for publication
Stepanov K N 1965 *Zh. Tekh. Fiz.* **35** 1002